The Gaussian curvature in special parameters.

Recall: Gauss and Weingarten equations:

- $\frac{\partial}{\partial u^k}\Gamma^s_{ij} - \frac{\partial}{\partial u^j}\Gamma^s_{ik} + \sum_r (\Gamma^r_{ij}\Gamma^s_{rk} - \Gamma^r_{ik}\Gamma^s_{rj}) = \sum_m (h_{ij}h_{km} - h_{ik}h_{jm}) g^{ms}$

- $\frac{\partial}{\partial u^k} h_{ij} - \frac{\partial}{\partial u^j} h_{ik} + \sum_r (\Gamma^r_{ij} h_{rk} - \Gamma^r_{ik} h_{rj}) = 0.$

For 2-dims surfaces in $\mathbb{R}^3$, the Gauss curvature depends only on the first fundamental form, (E, F, G).

In this section, we look for some explicit expressions of Gauss curvature in special parameters.

(1) Orthogonal parameters, lines of curvature parameters.

(Choose paratneter such that parameter lines are lines of curvature).

If $f: U \to \mathbb{R}^3$ has no umbilical points, choose local coordinate (u, v) s.t. $F = g_{12} = h_{12} = M = 0$, so

$$I = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}, \quad II = \begin{pmatrix} L & 0 \\ 0 & N \end{pmatrix}. \quad K_1 = \frac{L}{E}, \quad K_2 = \frac{N}{G}$$

Then:

$$L_v = \frac{E_v}{2}\left(\frac{L}{E} + \frac{N}{G}\right) = E_v \cdot H$$

$$N_u = \frac{G_u}{2}\left(\frac{L}{E} + \frac{N}{G}\right) = G_u \cdot H.$$

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right).$$

Proof:

$$2\Gamma_{ij,k} = g_{ik,j} + g_{jk,i} - g_{ij,k}$$

$$\Gamma_{ij}^{k} = \frac{1}{2} g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l})$$

Applying Codazzi-Mainardi equation for $i=j=1, k=2$.

$$0 = L_v - 0 + \Gamma_{11}^{2} h_{22} - \Gamma_{12}^{1} h_{11} = L_v - \frac{1}{2}\frac{E_v}{G}N - \frac{1}{2}\frac{E_v}{E}L$$

Since $\Gamma_{11}^{2} = \frac{1}{2} g^{22}(-g_{11,2}) = -\frac{1}{2}\frac{E_v}{G}$

$\Gamma_{12}^{1} = \frac{1}{2} g^{11}(g_{11,2}) = \frac{1}{2}\frac{E_v}{E}$

$(i=j=2, k=1)$

$$0 = N_u - 0 + \Gamma_{22}^{1} h_{11} - \Gamma_{12}^{2} h_{22} = N_u - \frac{1}{2}\frac{G_u}{E}L - \frac{1}{2}\frac{G_u}{G}N$$

Since $\Gamma_{22}^{1} = \frac{1}{2} g^{11}(-g_{22,1}) = -\frac{1}{2}\frac{G_u}{E}$

$\Gamma_{12}^{2} = \frac{1}{2} g^{22}(g_{22,1}) = \frac{1}{2}\frac{G_u}{G}$

$(i=j=1, k=\cancel{M}=2)$

$$\mathrm{Det}(\mathrm{I\!I}) = \sum_s \left((\Gamma_{11}^{s})_v - (\Gamma_{12}^{s})_u + \sum_r (\Gamma_{11}^{r}\Gamma_{r2}^{s} - \Gamma_{12}^{r}\Gamma_{r1}^{s}) \right) g_{s2}$$

$$= G\left(-\frac{1}{2}\left(\frac{E_v}{G}\right)_v - \frac{1}{2}\left(\frac{G_u}{G}\right)_u + \frac{1}{2}\frac{E_u}{E}\frac{1}{2}\frac{G_u}{G} - \frac{1}{2}\frac{E_v}{E}\left(-\frac{1}{2}\frac{E_v}{G}\right) + \left(-\frac{1}{2}\frac{E_v}{G}\right)\left(\frac{1}{2}\frac{G_v}{G}\right) - \frac{1}{2}\frac{G_u}{G}\frac{1}{2}\frac{G_u}{G} \right)$$

$$K = \frac{\mathrm{Det}(\mathrm{I\!I})}{\mathrm{Det}(\mathrm{I})}$$

$$= -\frac{1}{2EG}\left(E_{vv} - \frac{E_v \cdot G_v}{G} + G_{uu} - \frac{G_u^2}{G} - \frac{1}{2}\frac{E_u \cdot G_u}{E} + \frac{1}{2}\frac{E_v^2}{E} - \frac{E_v \cdot G_v}{2G} + \frac{G_u^2}{2G} \right)$$

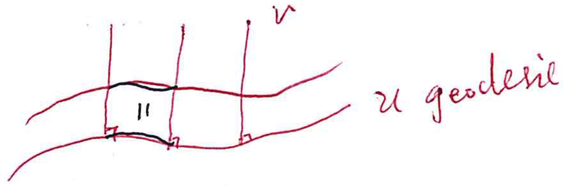
$$= -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u \right).$$

If $\mathrm{I} = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ isothermal parameter.

$$K = -\frac{1}{2\lambda}\left(\left(\frac{\lambda_v}{\lambda}\right)_v + \left(\frac{\lambda_u}{\lambda}\right)_u \right) = -\frac{1}{2\lambda}\Delta(\log\lambda).$$

(2) Geodesic parallel coordinates

Def: The u-curves ($v = \text{const}$) are geodesics parametrized by arc-length, which intersect each of v-curves orthogonally.



Rm:

1° Such coordinates occur iff $I = \begin{pmatrix} 1 & 0 \\ 0 & G \end{pmatrix}$.

$\Rightarrow$ by def.

$\Leftarrow$ For u-curves, $l = \int_0^u \sqrt{u_u^2 + u_v^2}\, du = u$.

So ~~pa~~ parametrized by arc-length.

$\nabla_{\partial u} \partial u = \Gamma_{11}^1 \partial u + \Gamma_{11}^2 \partial v = \frac{1}{2} g^{11}(-g_{11,1})\partial u + \frac{1}{2} g^{22}(g_{11,2})\partial v =$

u, v-orthogonal ($F = 0$).

So u-curves are geodesics.

2°. Specially, fix $u = u_{(0)}$, and constructing all the geodesics which are orthogonal to it.

If $u = u_{(0)}$ is geodesic, parametrized by arc-length.

Then $G(u^{(0)}, v) = 1$, $\frac{\partial}{\partial u} G(u^{(0)}, v) = 0$, $\Gamma_{ij}^k (u^{(0)}, v) = 0$.

One speaks of Fermi coordinate.

3°. By (1), we get for coordinate $\begin{pmatrix} 1 & 0 \\ 0 & G \end{pmatrix}$,

$$K(u, v) = -\frac{(\sqrt{G})_{uu}}{\sqrt{G}}$$

Since $K = -\frac{1}{2\sqrt{G}} \left(\frac{G_u}{\sqrt{G}} \right)_u = -\frac{1}{\sqrt{G}} \left(\frac{G_u}{2\sqrt{G}} \right)_u = -\frac{1}{\sqrt{G}} (\sqrt{G})_{uu}$.

(3) Example:

(1°) A surface of rotation $f(u,\varphi) = (r(u)\cos\varphi,\ r(u)\sin\varphi,\ h(u))$ with $r'^2 + h'^2 = 1$. It is parametrized by geodesic parallel coordinates because $I = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$.

These are Fermi coordinates in a neighborhood of a circle with $r' = 0$ and $r = 1$. $\left(\nabla_{\frac{\partial f}{\partial \varphi}} \frac{\partial f}{\partial \varphi} = -\frac{1}{2}(r^2)_u \frac{\partial f}{\partial \varphi} \right)$.

They have the same first fundamental form since in Fermi coordin

$f(t,\varphi) = \left(a\cos t \cos\frac{\varphi}{a},\ a\cos t \sin\frac{\varphi}{a},\ \int_0^t \sqrt{1 - a^2\sin^2 x}\, dx\right)$

around $t=0$, $I = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 t \end{pmatrix}$ is independent of a.

(check: $\nabla_{\frac{\partial f}{\partial \varphi}} \frac{\partial f}{\partial \varphi} = 0$ at $t=0$, $\nabla_{\frac{\partial f}{\partial t}} \frac{\partial f}{\partial t} = 0$.)

(2°) If $I = \begin{pmatrix} 1 & 0 \\ 0 & G^2 \end{pmatrix}$, $G = G(u,v)$. $ds^2 = du^2 + G(u,v)dv^2$

What can we say about G in terms of K_0? $K_0 = K|_0$.
Use polar coordinate (geodesic).
$G = u + \alpha_1 u^2 + \alpha_2 u^3 + o(u^3)$ $\left(\text{since } \lim_{u \to 0} \frac{\int_0^v \sqrt{G^2(u,v_1)}\,dv_1}{uv} = 1\right)$

$\frac{\partial^2 G}{\partial u^2}\Big|_0 = -K_0 G|_0$, $\quad \frac{\partial^3 G}{\partial u^3}\Big|_0 = -K_0\left(\frac{\partial G}{\partial u}\right)_0 - G\left(\frac{\partial K}{\partial u}\right)_0$

$2\alpha_1 = -K_0(u + \cdots)_0$, $\quad 6\alpha_2 = -K_0 - \left(\frac{\partial K}{\partial u}\right)_0 (u + \cdots)_0$

$\Rightarrow \alpha_1 = 0, \quad \alpha_2 = -\frac{1}{6}K_0$

$G = u - \frac{1}{6}K_0 u^3 + R_1(u,v)$.